A discontinuous Galerkin method for wave propagation in coupled elastic-acoustic media

Kaihang Guo[1], Sebastian Acosta[2], Jesse Chan[1]

[1]Department of Computational and Applied Mathematics, Rice University
[2]Department of Pediatrics-Cardiology, Baylor College of Medicine

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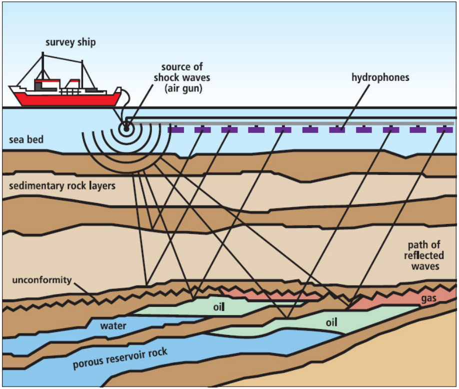
Motivation

In marine seismology, waves propagate through different subsurface layers, resulting in models with fluid-solid interfaces.

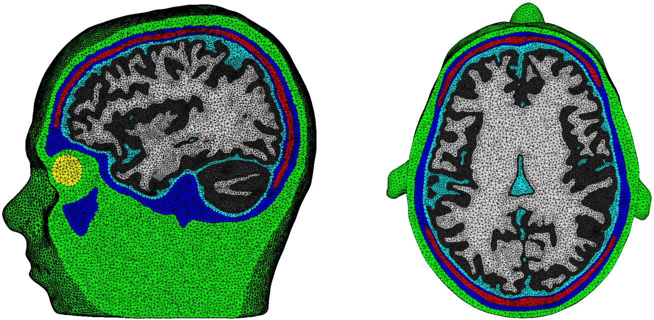


Marine seismic exploration

https://www.capeandislands.org/post/dynamite-going-your-bedroom-more-seismic-surveys-may-be-coming-atlantic-coast#stream/0

Motivation

In photoacoustic tomography (PAT), researchers want to locate brain tumors through reconstruction of initial pressure condition.



FEM mesh of an adult head

http://www.childbrain.eu/childbrain/keski-oikea/esrprojects/esr-13-development-of-new-finite-element-approaches-for-child-brain-research-and-comparison-to-standard-forward-modelling-methods-for-eeg-and-meg-source-analysis

Outline

- Elastic-acoustic coupled DG
- Numerical experiments
- Application examples

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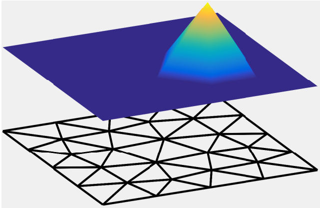
Finite element methods

Finite element methods (FEM):

- Unstructured meshes.
- Continuous piecewise polynomial approximation.



- Continuous PDE (example: advection)

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

- FEM weak form over domain Ω

$$\int_\Omega \frac{\partial u}{\partial t}\phi = \int_\Omega \frac{\partial u}{\partial x}\phi, \quad u, \phi \in V_h$$

Finite element methods

Finite element methods (FEM):

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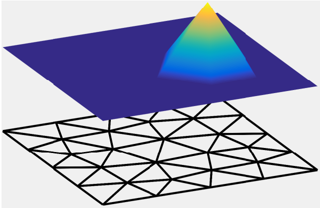
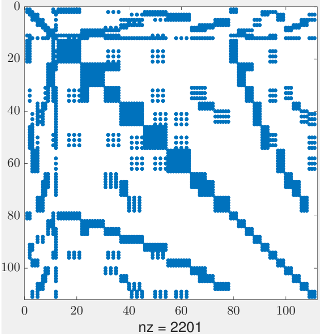




FEM yields system of ODEs with
global mass matrix $\mathbf{M}_\Omega$, discretization matrix $\mathbf{A}$.

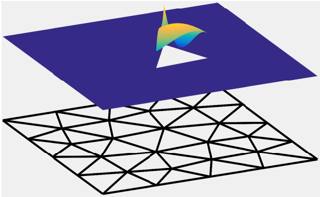
$$\mathbf{M}_\Omega \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{A}\mathbf{u}.$$

FEM mass matrix is globally coupled.

Basics of discontinuous Galerkin methods

Discontinuous Galerkin (DG) methods:

- Unstructured meshes.
- Weak continuity across faces.



- Continuous PDE (example: advection)

$$\frac{\partial u}{\partial t} = \frac{\partial f(u)}{\partial x}, \qquad f(u) = u.$$

- Local DG form with numerical flux $\boldsymbol{f}^*$: find $u \in P^N\left(D^k\right)$ such that

$$\int_{D_k} \frac{\partial u}{\partial t}\phi = \int_{D_k} \frac{\partial f(u)}{\partial x}\phi + \int_{\partial D_k} \boldsymbol{n} \cdot \left(\boldsymbol{f}^* - \boldsymbol{f}(u)\right)\phi, \qquad \forall \phi \in P^N\left(D^k\right).$$

Basics of discontinuous Galerkin methods

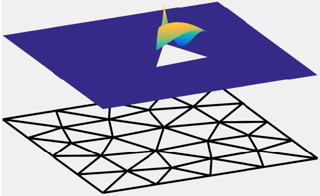
Discontinuous Galerkin (DG) methods:

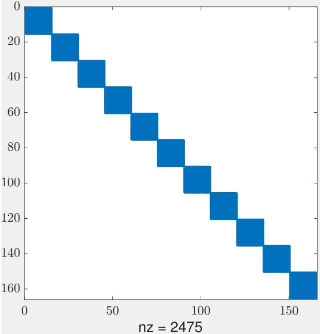
- Unstructured meshes.
- Weak continuity across faces.

DG in space yields system of ODEs

$$\mathbf{M}_\Omega \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{A}\mathbf{u}.$$

DG mass matrix decouples across elements, inter-element coupling only through $\mathbf{A}$.





Global mass matrix $\boldsymbol{M}_\Omega$.

High order DG methods

- Unstructured (tetrahedral) meshes for geometric flexibility.
- High order: low numerical dissipation and dispersion.
- High order approximations: more accurate per unknown.
- Explicit time stepping: high performance on many-core.

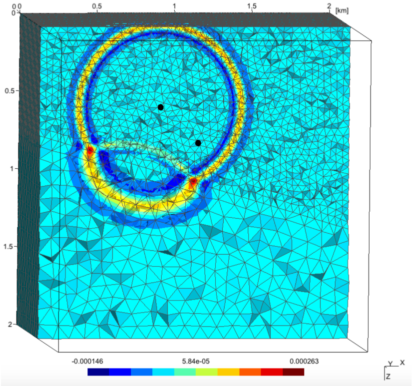


Figure courtesy of Axel Modave.

High order DG methods

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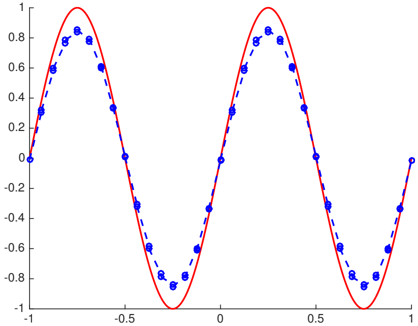

Fine linear approximation.

High order DG methods

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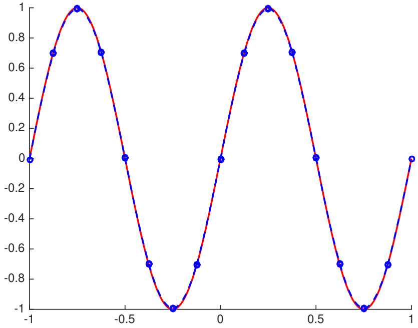

Coarse quadratic approximation.

High order DG methods

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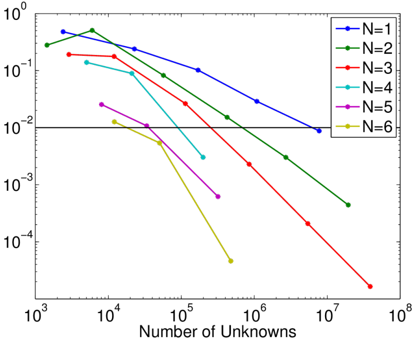

Max errors vs. dofs.

High order DG methods

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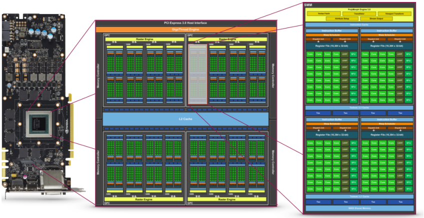


Graphics processing units (GPU).

Previous work

- Wilcox et al. constructed a DG-SEM scheme on quadrilateral and hexahedral meshes using Gauss-Lobatto quadrature by deriving an upwind numerical flux from the exact Riemann problem.[1]
- Zhan et al. extended this approach to anisotropic elastic-acoustic media by solving a simplified Riemann problem on each inter-element interface.[2]
- Ye et al. circumvent the Riemann problem altogether by using a DG formulation with a dissipative upwind-like “penalty” flux.[3]

Wilcox, Stadler, Burstedde, Ghattas. 2010. A high-order discontinuous Galerkin method for wave propagation through coupled elastic–acoustic media.

Zhan, Ren, Zhuang, Sun, Liu. 2018. An exact Riemann solver for wave propagation in arbitrary anisotropic elastic media with fluid coupling.

Ye, de Hoop, Petrovitch, Pyrak-Nolte, Wilcox. 2016. A discontinuous Galerkin method with a modified penalty flux for the propagation and scattering of acousto-elastic waves.

First-order wave equations

- Acoustic wave equation:

$$\frac{1}{c^2}\frac{\partial p}{\partial t} = \nabla \cdot \boldsymbol{u}, \qquad \frac{\partial \boldsymbol{u}}{\partial t} = \nabla p \qquad \text{(fluid)}$$

- Elastic wave equation:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = \sum_{i=1}^{d} \boldsymbol{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}_i}, \qquad \boldsymbol{C}^{-1}\frac{\partial \boldsymbol{\sigma}}{\partial t} = \sum_{i=1}^{d} \boldsymbol{A}_i \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_i} \qquad \text{(solid)}$$

- Numerical scheme: high order, explicit time-stepping, parallelizable

Strong DG formulation

- Pure acoustic domain:

$$\left(\frac{1}{c^2}\frac{\partial p}{\partial t}, q\right)_{L^2(D^k)} = (\nabla \cdot \boldsymbol{u}, q)_{L^2(D^k)} + \sum_{f \in \partial D^k \cap \Gamma_{aa}} \left\langle \frac{1}{2}\boldsymbol{n}^T [\![\boldsymbol{u}]\!] + \frac{\tau_p}{2}[\![p]\!], q \right\rangle_{L^2(f)}$$

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{w}\right)_{L^2(D^k)} = (\nabla p, \boldsymbol{w})_{L^2(D^k)} + \sum_{f \in \partial D^k \cap \Gamma_{aa}} \left\langle \frac{1}{2}\boldsymbol{n}^T [\![p]\!] + \frac{\tau_u}{2}[\![\boldsymbol{u}]\!], \boldsymbol{w} \right\rangle_{L^2(f)}$$

- Pure elastic domain:

$$\left(\rho\frac{\partial \boldsymbol{v}}{\partial t}, \boldsymbol{w}\right)_{L^2(D^k)} = \left(\sum_{i=1}^{d} \boldsymbol{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}_i}, \boldsymbol{w}\right)_{L^2(D^k)} + \sum_{f \in \partial D^k \cap \Gamma_{ee}} \left\langle \frac{1}{2}\boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!] + \frac{\tau_v}{2}\boldsymbol{A}_n^T \boldsymbol{A}_n [\![\boldsymbol{v}]\!], \boldsymbol{w} \right\rangle_{L^2(f)}$$

$$\left(\boldsymbol{C}^{-1}\frac{\partial \boldsymbol{\sigma}}{\partial t}, \boldsymbol{q}\right)_{L^2(D^k)} = \left(\sum_{i=1}^{d} \boldsymbol{A}_i \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_i}, \boldsymbol{q}\right)_{L^2(D^k)} + \sum_{f \in \partial D^k \cap \Gamma_{ee}} \left\langle \frac{1}{2}\boldsymbol{A}_n [\![\boldsymbol{v}]\!] + \frac{\tau_\sigma}{2}\boldsymbol{A}_n \boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!], \boldsymbol{q} \right\rangle_{L^2(f)}$$

Γ_{aa} : acoustic-acoustic interfaces $\quad$ Γ_{ee} : elastic-elastic interfaces

Weight-adjusted DG: stable, accurate, non-invasive

- High order wavespeeds: weighted mass matrices. Stable, but requires pre-computation/storage of inverses or factorizations!

$$\boldsymbol{M}_{1/c^2}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{A}_h\boldsymbol{U}, \qquad \left(\boldsymbol{M}_{1/c^2}\right)_{ij} = \int_{D^k}\frac{1}{c^2(\boldsymbol{x})}\phi_j(\boldsymbol{x})\phi_i(\boldsymbol{x}).$$

- Weight-adjusted DG (WADG): energy stable approx. of $\boldsymbol{M}_{1/c^2}$

$$\boldsymbol{M}_{1/c^2} \approx \boldsymbol{M}\left(\boldsymbol{M}_{c^2}\right)^{-1}\boldsymbol{M} \;\Rightarrow\; \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{M}^{-1}\left(\boldsymbol{M}_{c^2}\right)\boldsymbol{M}^{-1}\boldsymbol{A}_h\boldsymbol{U}$$

- Low storage matrix-free application of $\boldsymbol{M}^{-1}\boldsymbol{M}_{c^2}$ using **quadrature**-based interpolation and L^2 projection matrices $\boldsymbol{V}_q, \boldsymbol{P}_q$.

$$\left(\boldsymbol{M}\right)^{-1}\boldsymbol{M}_{c^2}\mathrm{RHS} = \underbrace{\boldsymbol{M}^{-1}\boldsymbol{V}_q^T W}_{\boldsymbol{P}_q}\operatorname{diag}\left(c^2\right)\boldsymbol{V}_q\left(\mathrm{RHS}\right).$$

Chan, Hewett, Warburton. 2016. Weight-adjusted DG methods: wave propagation in heterogeneous media.

Energy stable elastic-acoustic coupling

- Typical DG approach: upwind flux (exact Riemann solver).
- Riemann problem is expensive and difficult to solve exactly in heterogeneous and anisotropic media.
- The numerical flux should be consistent with continuity conditions on elastic-acoustic interfaces

$$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \boldsymbol{n}, \qquad \boldsymbol{A}_n^T \boldsymbol{\sigma} = p\boldsymbol{n}.$$

- Penalty term with parameter $\tau \geq 0$ adds upwind-like dissipation.
- Our goal is to find a numerical flux such that the DG scheme is energy stable.

Energy stable elastic-acoustic coupling

(Elastic)

$$\frac{1}{2}\left\langle p\boldsymbol{n} - \boldsymbol{A}_n^T\boldsymbol{\sigma} - (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^T)\boldsymbol{A}_n^T\boldsymbol{\sigma}, \boldsymbol{w}\right\rangle + \frac{\tau}{2}\left\langle(\boldsymbol{u} - \boldsymbol{v})\cdot\boldsymbol{n}, \boldsymbol{w}\cdot\boldsymbol{n}\right\rangle$$

$$\frac{1}{2}\left\langle(\boldsymbol{u} - \boldsymbol{v})\cdot\boldsymbol{n}, \boldsymbol{A}_n^T\boldsymbol{q}\right\rangle + \frac{\tau}{2}\left\langle(p\boldsymbol{n} - \boldsymbol{A}_n^T\boldsymbol{\sigma}), \boldsymbol{A}_n^T\boldsymbol{q}\right\rangle$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{v}\cdot\boldsymbol{n}$$

$$\boldsymbol{A}_n^T\boldsymbol{\sigma} = p\boldsymbol{n}$$

$$\frac{1}{2}\left\langle(\boldsymbol{A}_n^T\boldsymbol{\sigma} - p\boldsymbol{n})\cdot\boldsymbol{n}, \boldsymbol{w}\cdot\boldsymbol{n}\right\rangle + \frac{\tau}{2}\left\langle(\boldsymbol{v} - \boldsymbol{u})\cdot\boldsymbol{n}, \boldsymbol{w}\cdot\boldsymbol{n}\right\rangle$$

$$\frac{1}{2}\left\langle(\boldsymbol{v} - \boldsymbol{u})\cdot\boldsymbol{n}, q\right\rangle + \frac{\tau}{2}\left\langle(\boldsymbol{A}_n^T\boldsymbol{\sigma} - p\boldsymbol{n})\cdot\boldsymbol{n}, q\right\rangle$$

(Acoustic)

Theoretical results

Theorem (Consistency)

The coupled discontinuous Galerkin scheme is consistent.

Theorem (Energy stability)

The coupled discontinuous Galerkin scheme is energy stable for $\tau_u = \tau_v \geq 0, \tau_p = \tau_\sigma \geq 0$, *in the sense that*

$$\sum_{D^k \in \Omega_h^e} \frac{\partial}{\partial t} \left((\rho \boldsymbol{v}, \boldsymbol{v})_{L^2(D^k)} + \left(\boldsymbol{C}^{-1} \boldsymbol{\sigma}, \boldsymbol{\sigma} \right)_{L^2(D^k)} \right) + \sum_{D^k \in \Omega_h^a} \frac{\partial}{\partial t} \left(\left(\frac{p}{c^2}, p \right)_{L^2(D^k)} + (\boldsymbol{u}, \boldsymbol{u})_{L^2(D^k)} \right)$$

$$= - \sum_{f \in \Gamma_{aa}} \int_f \left(\tau_p [\![p]\!]^2 + \tau_u \left(\boldsymbol{n} \cdot [\![\boldsymbol{u}]\!] \right)^2 \right) - \sum_{f \in \Gamma_{ee}} \int_f \left(\frac{\tau_u}{2} \left| \boldsymbol{A}_n [\![\boldsymbol{v}]\!] \right|^2 + \frac{\tau_p}{2} \left| \boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!] \right|^2 \right)$$

$$- \sum_{f \in \Gamma_{ea} \cup \Gamma_{ae}} \int_f \left(\frac{\tau_u}{2} \left| \boldsymbol{n}^T (\boldsymbol{u} - \boldsymbol{v}) \right|^2 + \frac{\tau_p}{2} \left| p\boldsymbol{n} - \boldsymbol{A}_n^T \boldsymbol{\sigma} \right|^2 \right) \leq 0,$$

where Ω_h^a *and* Ω_h^e *denote the acoustic and elastic computational domain, respectively.*

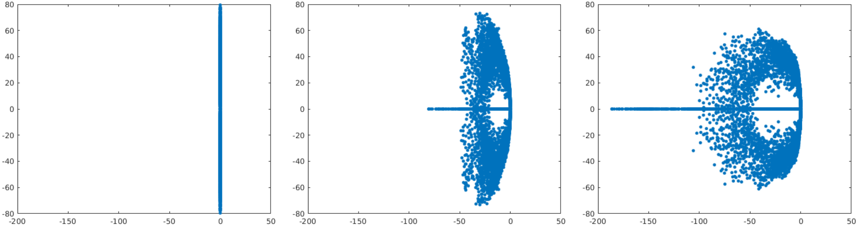
Outline

- Elastic-acoustic coupled DG
- Numerical experiments
- Application examples

Spectra and choice of penalty parameter

Let $\boldsymbol{L}$ denote the matrix induced by the global semi-discrete DG formulation, such that the time evolution of the global solution is governed by

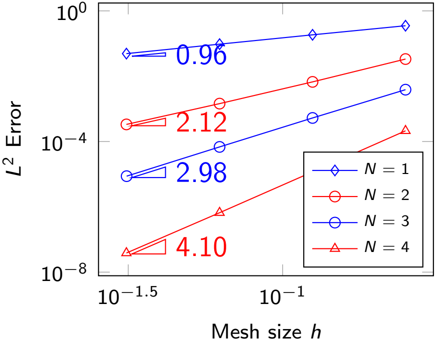
$$\frac{\partial \boldsymbol{Q}}{\partial t} = \boldsymbol{L}\boldsymbol{Q}$$



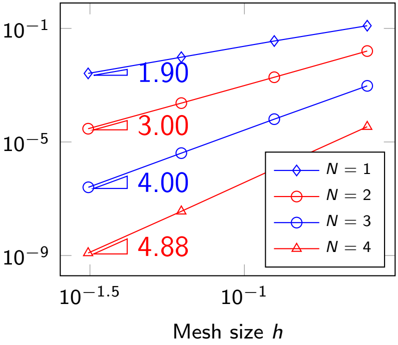
(a) $\tau_u = \tau_p = 0$ (b) $\tau_u = \tau_p = \frac{1}{2}$ (c) $\tau_u = \tau_p = 1$

Figure: Spectra for $N = 3$ on a non-curved uniform mesh with $h = 1/4$. For all cases, the largest real part of the spectra is $O(10^{-14})$.

Classical interface problems: Scholte wave



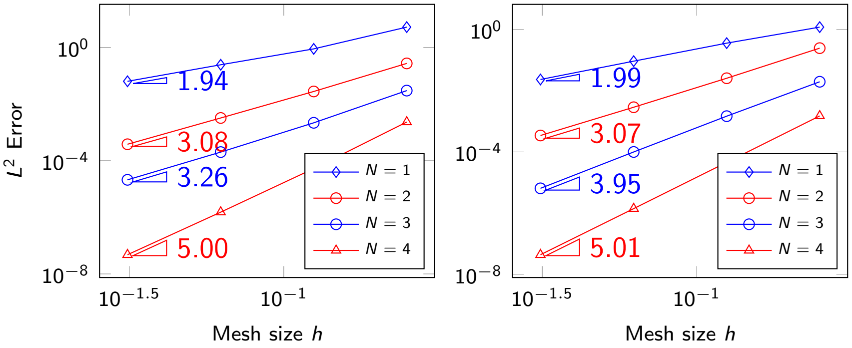
(a) Central flux ($\tau_u = \tau_p = 0$)



(b) Penalty flux ($\tau_u = \tau_p = 1$)

Figure: Convergence of L^2 errors for the Scholte wave solution

Classical interface problems: Snell's law

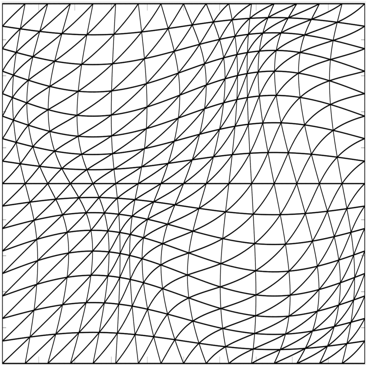


(a) Central flux ($\tau_u = \tau_p = 0$)

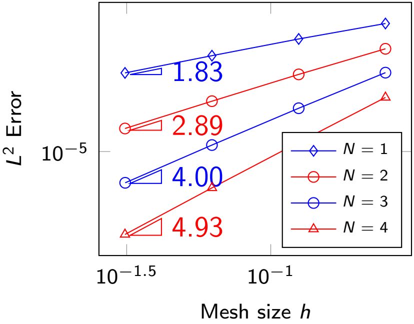
(b) Penalty flux ($\tau_u = \tau_p = 1$)

Figure: Convergence of L^2 errors for the Snell's law solution

Extension to curvilinear meshes

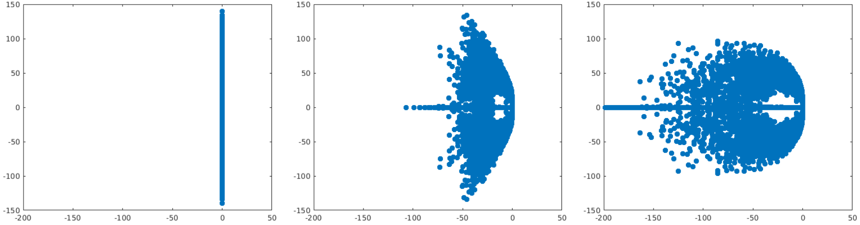


(a) Curvilinear mesh



(b) Scholte wave (curvilinear)

Extension to curvilinear meshes



(a) $\tau_u = \tau_p = 0$ (b) $\tau_u = \tau_p = \frac{1}{2}$ (c) $\tau_u = \tau_p = 1$

Figure: Spectra of the discontinuous Galerkin discretization matrix for central and penalty fluxes on a warped curvilinear mesh of degree $N = 3$. For all cases, the largest real part of the spectra is $O(10^{-14})$.

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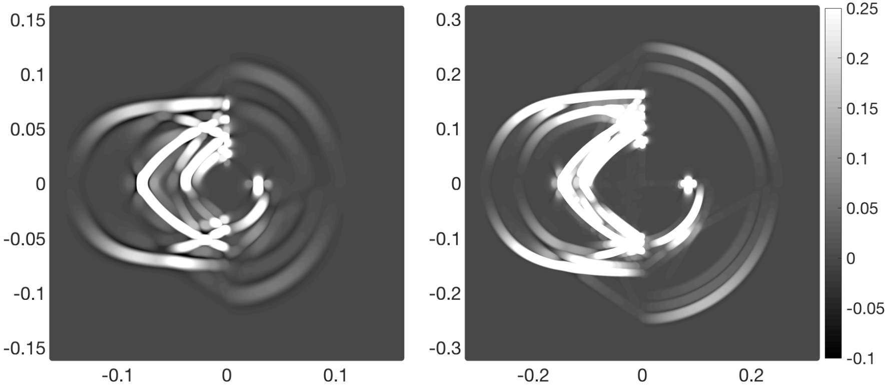
Homogeneous anisotropic media

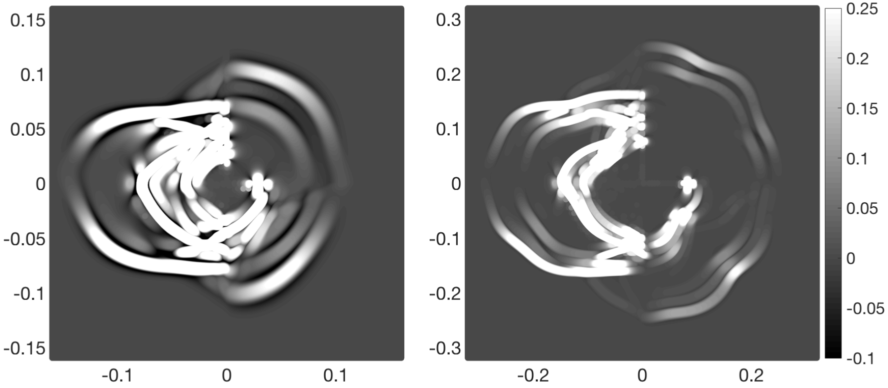


(a) $T = 30$ (b) $T = 60$

Figure: An example of wave propagation in homogeneous anisotropic-isotropic acoustic-elastic media.

Komatitsch, Barnes, Tromp. 2000. Simulation of anisotropic wave propagation based upon a spectral element method.

Heterogeneous anisotropic media

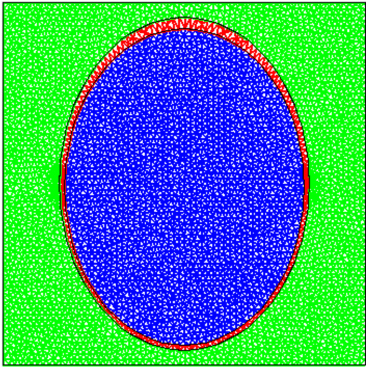


(a) $T = 30$ (b) $T = 60$

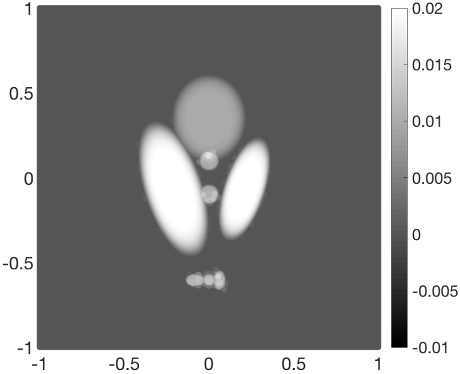
Figure: An example of wave propagation in heterogeneous anisotropic-isotropic acoustic-elastic media.

Komatitsch, Barnes, Tromp. 2000. Simulation of anisotropic wave propagation based upon a spectral element method.

Photoacoustic tomography (PAT)

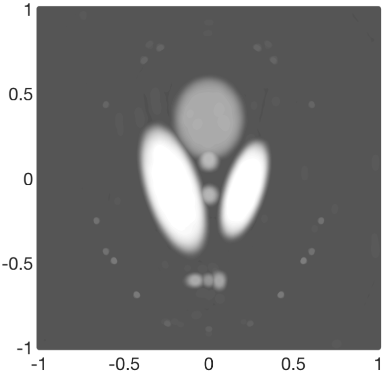


(a) Mesh

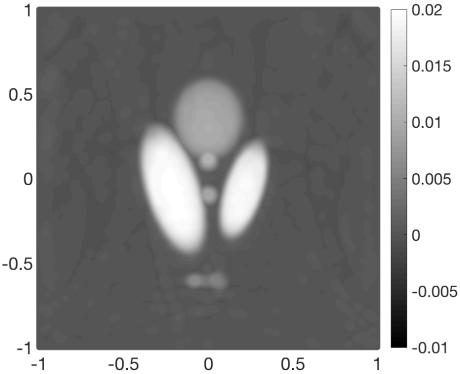


(b) Exact initial pressure

Photoacoustic tomography (PAT)



(a) Reconstruction after 5 iterations

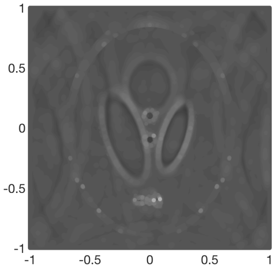


(b) Purely acoustic reconstruction

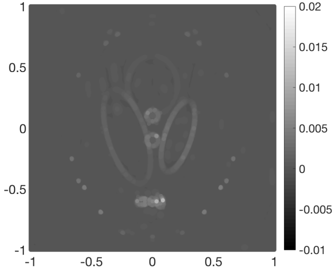
Photoacoustic tomography (PAT)

Iteration	Fine	Fine (acous)	Coarse	Coarse (acous)
1	0.140530	0.147435	0.140556	0.147103
2	0.094658	0.133881	0.094811	0.133508
3	0.075081	0.130397	0.075347	0.130010
4	0.065585	0.129331	0.065941	0.128939
5	0.060577	0.128973	0.060998	0.128577

Table: Relative L^2 errors at each iteration



(a) Error after 1 iter



(b) Error after 5 iter

Summary and acknowledgements

- We derive a numerical flux across elastic-acoustic interfaces with a very simple form.
- The proposed scheme can be applied on unstructured tetrahedral meshes and general curvilinear meshes.
- The resulting DG method is efficient, provably energy stable, and high order accurate for arbitrary heterogeneous and anisotropic media.

Thank you! Questions?

Guo, Acosta, Chan. 2019. A weight-adjusted DG method for wave propagation in coupled elastic-acoustic media.

Guo, Chan. 2018. Bernstein-Bezier weight-adjusted discontinuous Galerkin methods for wave propagation(JCP).

Chan, Hewett, Warburton. 2016. Weight-adjusted DG methods: wave propagation in heterogeneous media (SISC).

Chan. 2017. Weight-adjusted DG methods: matrix-valued weights and elastic wave prop. in heterogeneous media (IJNME).

Chan, Warburton. 2015. GPU-accelerated Bernstein-Bezier discontinuous Galerkin methods for wave propagation (SISC).