Weight-adjusted Discontinuous Galerkin Methods on Moving Curved Meshes

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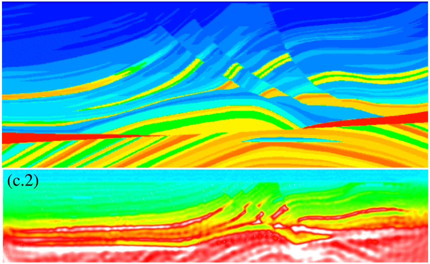
Numerical simulation of wave propagation

Many procedures require **accurately** and **efficiently** solving hyperbolic partial differential equations (PDEs) in realistic settings.

- Seismic and medical imaging
- Engineering design
- Computational fluids



S. Scott Collis et al. Unstructured Discontinuous Galerkin for Seismic Inversion (Sandia).

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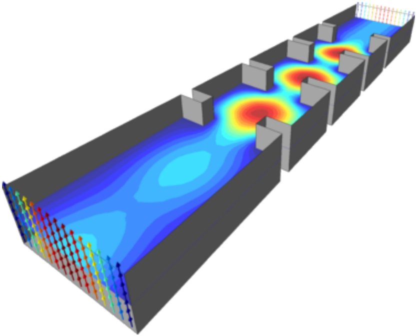

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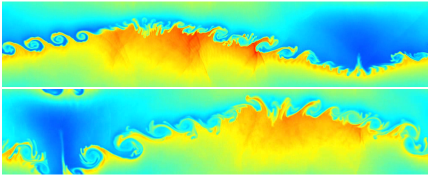

Per-Olof Persson, http://persson.berkeley.edu/research.html.

Discontinuous Galerkin (DG) methods for waves

- Unstructured (tetrahedral) meshes for geometric flexibility.
- High order: low numerical dissipation and dispersion.
- High order approximations: more accurate per unknown.

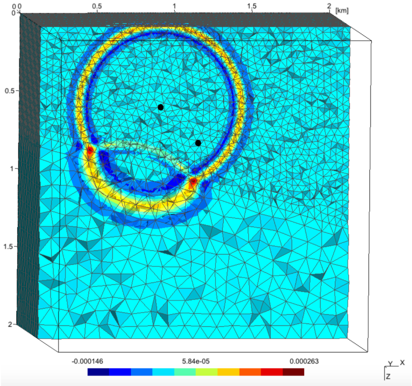

Figure courtesy of Axel Modave.

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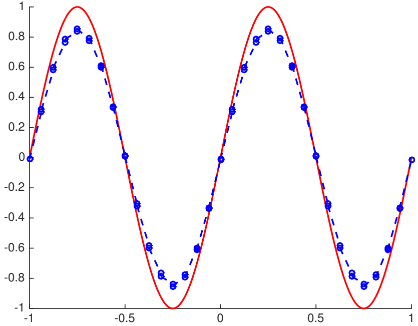

Fine linear approximation.

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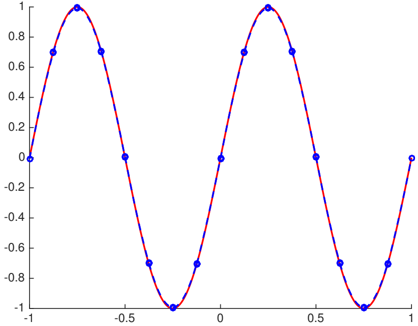

Coarse quadratic approximation.

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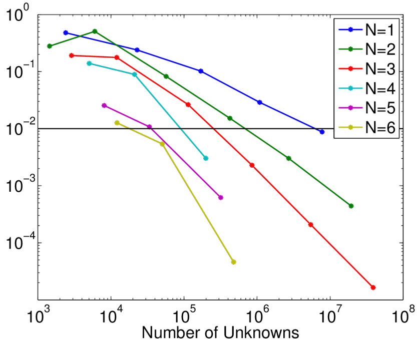
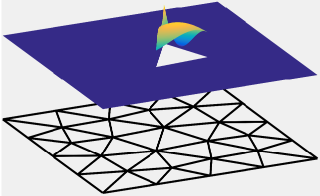


Max errors vs. dofs.

Discontinuous Galerkin (DG) methods:

- Piecewise polynomial approximation.
- Weak continuity across faces.



- Continuous PDE (example: advection)

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

- DG local strong form over D_k with numerical flux $\boldsymbol{f}^*$.

$$\int_{D_k} \frac{\partial u}{\partial t}\phi = \int_{D_k} \frac{\partial u}{\partial x}\phi + \int_{\partial D_k} \boldsymbol{n}\cdot\left(\boldsymbol{f}^* - \boldsymbol{f}(u)\right)\phi, \qquad u,\phi \in V_h$$

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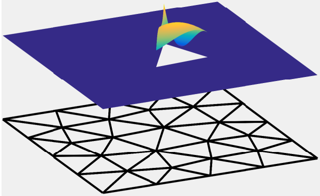
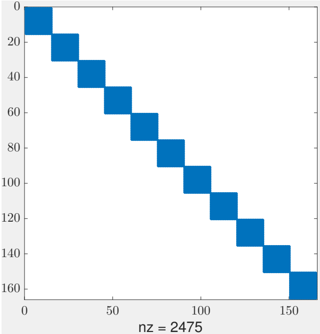


DG yields system of ODEs

$$\mathbf{M}_{\Omega}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{A}\mathbf{u}.$$

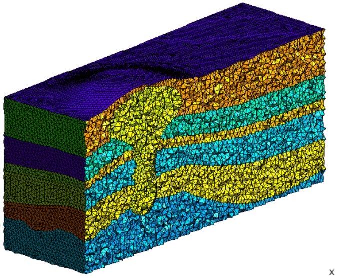
DG mass matrix decouples across elements, inter-element coupling only through $\mathbf{A}$.



Time-domain nodal DG methods

Assume $u(\boldsymbol{x}, t) = \sum \boldsymbol{u}_j \phi_j(\boldsymbol{x})$ on D^k

- Compute numerical flux at face nodes (non-local).
- Compute RHS of (local) ODE.
- Evolve (local) solution using explicit time integration (RK, AB, etc).



Mesh courtesy of J.F. Remacle

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

Example: advection equation.

$$\boldsymbol{M}_{ij} = \int_{D^k} \phi_j(\boldsymbol{x}) \phi_i(\boldsymbol{x})$$

$$\mathbf{L}_f = \mathbf{M}^{-1} \mathbf{M}_f.$$

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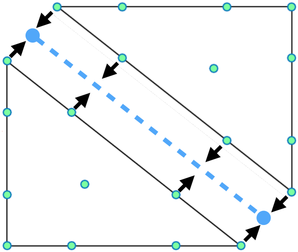

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{D}_x \mathbf{u} + \sum_{\text{faces}} \mathbf{L}_f \left(\text{flux}\right).$$

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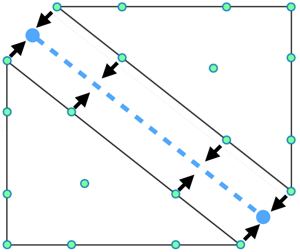

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \underbrace{\mathbf{D}_x \mathbf{u}}_{\text{Volume kernel}} + \underbrace{\sum_{\text{faces}} \mathbf{L}_f \left(\text{flux}\right)}_{\text{Surface kernel}}.$$

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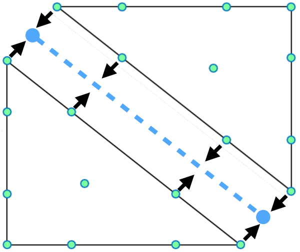

$$\underbrace{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}}_{\text{Update kernel}} = \underbrace{\mathbf{D}_x \mathbf{u}}_{\text{Volume kernel}} + \underbrace{\sum_{\text{faces}} \mathbf{L}_f \left(\text{flux}\right)}_{\text{Surface kernel}}.$$

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Outline

1. Weight-adjusted DG (WADG): high order heterogeneous media

2. Arbitrary Lagrangian-Eulerian DG: moving meshes

Outline

Energy stable discontinuous Galerkin formulations

- Model problem: acoustic wave equation (pressure-velocity system)

$$\frac{1}{c^2}\frac{\partial p}{\partial t} = \nabla \cdot \boldsymbol{u}, \qquad \frac{\partial \boldsymbol{u}}{\partial t} = \nabla p.$$

- Jumps of solutions:

$$[\![p]\!] = p^+ - p, \qquad [\![\boldsymbol{u}]\!] = \boldsymbol{u}^+ - \boldsymbol{u}.$$

- Local formulation

$$\int_{D^k} \frac{1}{c^2}\frac{\partial p}{\partial t} q = \int_{D^k} \nabla \cdot \boldsymbol{u} q + \frac{1}{2}\int_{\partial D^k} \left([\![\boldsymbol{u}]\!] \cdot \boldsymbol{n} + \tau_p [\![p]\!]\right) q,$$

$$\int_{D^k} \frac{\partial \boldsymbol{u}}{\partial t} \boldsymbol{v} = \int_{D^k} \nabla p \cdot \boldsymbol{v} + \frac{1}{2}\int_{\partial D^k} \left([\![p]\!] + \tau_u [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}\right) \boldsymbol{v}.$$

- High order accuracy, semi-discrete energy stability

$$\frac{\partial}{\partial t}\left(\sum_k \int_{D^k} \frac{p^2}{c^2} + |\boldsymbol{u}|^2\right) = -\sum_k \int_{\partial D^k} \tau_p [\![p]\!]^2 + \tau_u [\![\boldsymbol{u} \cdot \boldsymbol{n}]\!]^2 \leq 0.$$

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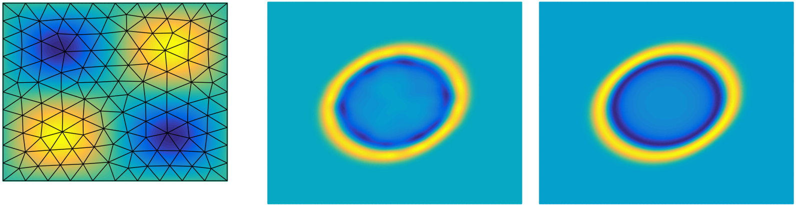
High order approximation of smoothly varying media



(a) Mesh and exact c^2 (b) Piecewise const. c^2 (c) High order c^2

- Piecewise const. c^2: energy stable and efficient, but inaccurate.

$$\frac{1}{c^2(\boldsymbol{x})}\frac{\partial p}{\partial t} + \nabla \cdot \boldsymbol{u} = 0, \qquad \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p = 0.$$

- High order wavespeeds: weighted mass matrices. Stable, but expensive (pre-computation + storage of matrix inverses)!

$$\boldsymbol{M}_{1/c^2}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{A}_h\boldsymbol{U}, \qquad \left(\boldsymbol{M}_{1/c^2}\right)_{ij} = \int_{D^k}\frac{1}{c^2(\boldsymbol{x})}\phi_j(\boldsymbol{x})\phi_i(\boldsymbol{x}).$$

Weight-adjusted DG (WADG)

- Weight-adjusted DG: provably energy stable approx. of $\boldsymbol{M}_{1/c^2}$

$$\boldsymbol{M}_{1/c^2}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \approx \boldsymbol{M}\left(\boldsymbol{M}_{c^2}\right)^{-1}\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{A}_h\boldsymbol{U}.$$

- New evaluation reuses implementation for constant wavespeed

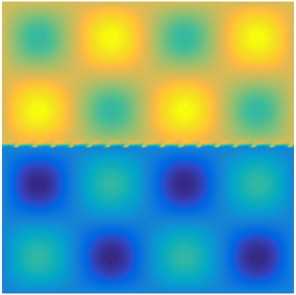
$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \underbrace{\boldsymbol{M}^{-1}\left(\boldsymbol{M}_{c^2}\right)}_{\text{modified update}} \qquad \underbrace{\boldsymbol{M}^{-1}\boldsymbol{A}_h\boldsymbol{U}}_{\text{constant wavespeed RHS}}$$

- Low storage matrix-free application of $\boldsymbol{M}^{-1}\boldsymbol{M}_{c^2}$ using **quadrature**-based interpolation and L^2 projection matrices $\boldsymbol{V}_q, \boldsymbol{P}_q$.

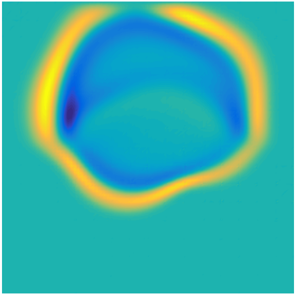
$$\left(\boldsymbol{M}\right)^{-1}\boldsymbol{M}_{c^2} = \underbrace{\boldsymbol{M}^{-1}\boldsymbol{V}_q^T\boldsymbol{W}}_{\boldsymbol{P}_q}\operatorname{diag}\left(c^2\right)\boldsymbol{V}_q.$$

Chan, Hewett, Warburton (2017). Weight-adjusted DG methods: wave propagation in heterogeneous media.

WADG: nearly identical to DG w/weighted mass matrices



(a) $c^2(x, y)$



(b) Standard DG

Figure: Standard vs. weight-adjusted DG with spatially varying c^2.

- The L^2 error is $O(h^{N+1})$, but the difference between the DG and WADG solutions is $O(h^{N+2})$!

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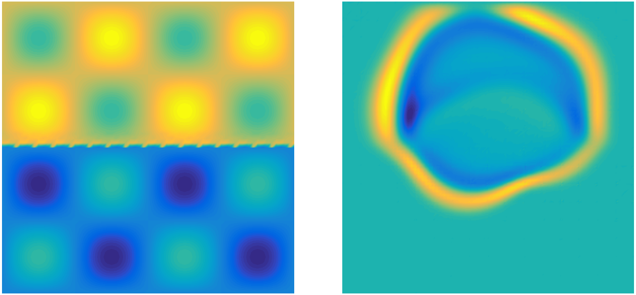

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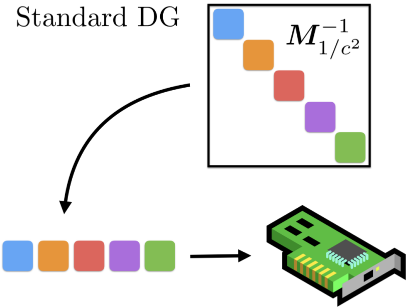
WADG: more efficient than storing M^{-1}_{1/c^2} on GPUs

	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$
DG	.66	2.79	9.90	29.4	73.9	170.5	329.4
WADG	0.59	1.44	4.30	13.9	43.0	107.8	227.7
Speedup	1.11	1.94	2.30	2.16	1.72	1.58	1.45

Time (ns) per element: storing/applying $\boldsymbol{M}^{-1}_{1/c^2}$ vs WADG (deg. $2N$ quadrature).



Efficiency on GPUs: reduce memory accesses and data movement!

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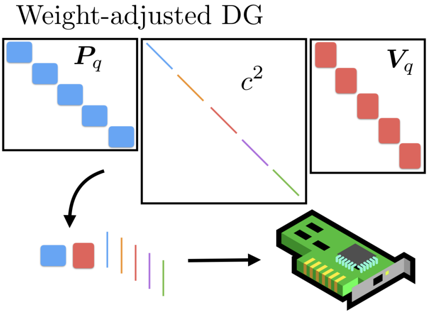

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Outline

Efficient way to capture domain movement

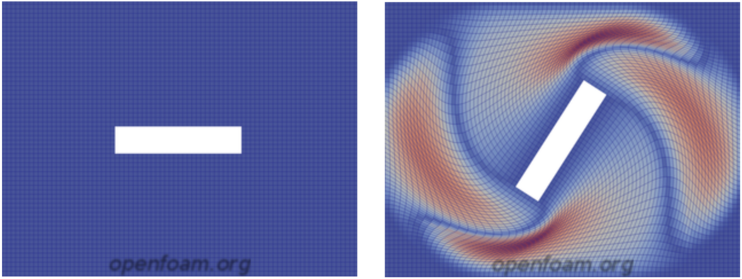


Figure: Rotating bar: an example of a moving domain

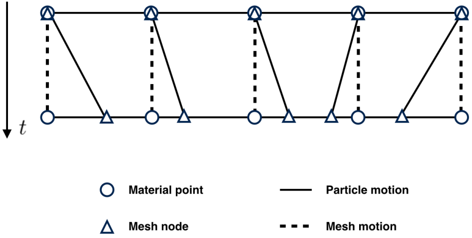
Simulations on moving domains require moving mesh methods.

https://openfoam.org/release/2-3-0/mesh-motion/

Arbitrary Lagrangian-Eulerian (ALE) framework

ALE combines advantages of Lagrangian and Eulerian formulations.

- Eulerian methods: mesh fixed in space
- Lagrangian methods: mesh must be evolved along with the solution
- ALE methods: mesh can move arbitrarily



Eulerian description

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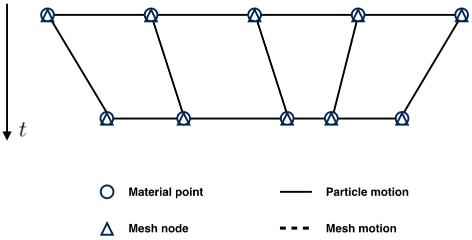

Lagrangian description

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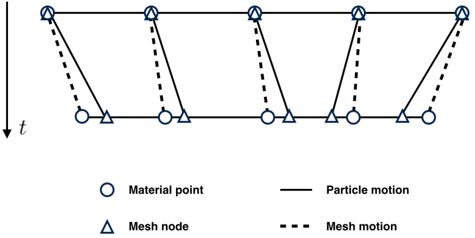
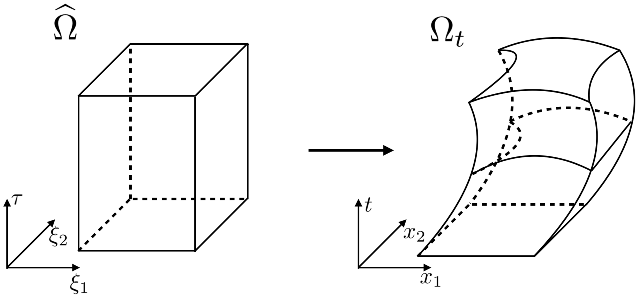


Arbitrary Lagrangian-Eulerian description

ALE transformation



- ALE tranformation:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \sum_j \frac{\partial \xi_j}{\partial t} \frac{\partial}{\partial \xi_j},$$

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j}, \qquad i = 1, 2.$$

ALE formulation of a conservation law

- Conservation law on the moving physical domain:

$$\frac{d\boldsymbol{q}}{dt} + \nabla \cdot \boldsymbol{f} = 0.$$

- Conservation law on the stationary reference domain:

$$\frac{d\boldsymbol{q}J}{d\tau} + \sum_j \frac{\partial \xi_j}{\partial t}\frac{\partial \boldsymbol{q}J}{\partial \xi_j} + \sum_i \sum_j \frac{\partial \xi_j}{\partial x_i}\frac{\partial J\boldsymbol{f}_i}{\partial \xi_j} = 0.$$

Additional geometric conservation law: $\frac{\partial J}{\partial \tau} + \widehat{\nabla} \cdot (J\widehat{x}_t) = 0.$

Energy stable skew-symmetric ALE-DG

- Constant solution on a moving mesh:

$$\frac{\partial u}{\partial t} = 0.$$

- ALE system on a stationary reference mesh:

$$\frac{\partial uJ}{\partial \tau} + \widehat{\nabla} \cdot (uJ\widehat{x}_t) = 0,$$
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- ALE-DG formulation:

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$$\begin{aligned} \frac{\partial uJ}{\partial \tau} + \widehat{\nabla} \cdot (uJ\widehat{x}_t) &= 0, \\ \frac{\partial J}{\partial \tau} + \widehat{\nabla} \cdot (J\widehat{x}_t) &= 0. \end{aligned}$$

- ALE-DG formulation:

$$\begin{aligned} \left(\frac{\partial uJ}{\partial \tau}, v\right) + \left(\widehat{\nabla} \cdot (uJ\widehat{x}_t), v\right) &= 0, \\ \left(\frac{\partial J}{\partial \tau}, w\right) + \left(\widehat{\nabla} \cdot (J\widehat{x}_t), w\right) &= 0. \end{aligned}$$

Energy stable skew-symmetric ALE-DG

- Constant solution on a moving mesh:

$$\frac{\partial u}{\partial t} = 0.$$

- ALE system on a stationary reference mesh:

$$\begin{aligned}\frac{\partial uJ}{\partial \tau} + \widehat{\nabla} \cdot (uJ\widehat{x}_t) = 0,\\ \frac{\partial J}{\partial \tau} + \widehat{\nabla} \cdot (J\widehat{x}_t) = 0.\end{aligned}$$

- ALE-DG formulation:

$$\begin{aligned}\left(\frac{\partial {\color{red}uJ}}{\partial \tau}, v\right) + \left(\widehat{\nabla} \cdot (uJ\widehat{x}_t), v\right) = 0,\\ \left(\frac{\partial {\color{red}J}}{\partial \tau}, w\right) + \left(\widehat{\nabla} \cdot (J\widehat{x}_t), w\right) = 0.\end{aligned}$$

Energy stable skew-symmetric ALE-DG

- Skew-symmetric ALE-DG formulation:

$$\left(\frac{\partial uJ}{\partial \tau}, v\right) + \frac{1}{2}\left(\widehat{\nabla}\cdot\left(J\widehat{x}_t\right)u, v\right)$$
$$+\frac{1}{2}\left\{\left(\widehat{\nabla}\cdot\left(uJ\widehat{x}_t\right), v\right) + \langle n\cdot u^+, J\widehat{x}_t v\rangle - \left(u, \widehat{\nabla}\cdot\left(J\widehat{x}_t v\right)\right)\right\} = 0,$$
$$\left(\frac{\partial J}{\partial \tau}, w\right) + \left(\widehat{\nabla}\cdot\left(J\widehat{x}_t\right), w\right) = 0.$$

- Skew-symmetric term:

Energy stable skew-symmetric ALE-DG

- Skew-symmetric ALE-DG formulation:

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Energy stable skew-symmetric ALE-DG

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$$\left(\frac{\partial uJ}{\partial \tau}, v\right) + \frac{1}{2}\left(\widehat{\nabla}\cdot(J\widehat{x}_t)\,u, v\right)$$
$$+\frac{1}{2}\left\{\left(\widehat{\nabla}\cdot(uJ\widehat{x}_t), v\right) + \langle n\cdot u^{+}, J\widehat{x}_t v\rangle - \left(u, \widehat{\nabla}\cdot(J\widehat{x}_t v)\right)\right\} = 0,$$
$$\left(\frac{\partial J}{\partial \tau}, w\right) + \left(\widehat{\nabla}\cdot(J\widehat{x}_t), w\right) = 0.$$

- Skew-symmetric term:

$$S(u,v) = \sum \frac{1}{2}\left\{\left(\widehat{\nabla}\cdot(uJ\widehat{x}_t), v\right) + \langle n\cdot u^{+}, J\widehat{x}_t v\rangle - \left(u, \widehat{\nabla}\cdot(J\widehat{x}_t v)\right)\right\}$$

Energy stable skew-symmetric ALE-DG

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Energy conservation (DG)

- In DG methods, solution u is related to uJ through:

$$(u, vJ) = (uJ, v) \iff \boldsymbol{M}_J u = \boldsymbol{M}(uJ) \iff u = \boldsymbol{M}_J^{-1}\boldsymbol{M}(uJ)$$

- Summing over elements:

- L^2 projection preserves polynomial moments:

$$\sum \frac{1}{2}\left(\frac{\partial J}{\partial \tau}, u^2\right) + \sum \frac{1}{2}\left(\widehat{\nabla}\cdot\left(J\widehat{x}_t\right), u^2\right) = 0.$$

- Subtracting these two equations gives

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$$\sum \left(\frac{\partial uJ}{\partial \tau}, v\right) + \sum \frac{1}{2}\left(\widehat{\nabla} \cdot (J\widehat{x}_t)\, u, v\right) + S(u, v) = 0,$$
$$\sum \left(\frac{\partial J}{\partial \tau}, w\right) + \sum \left(\widehat{\nabla} \cdot (J\widehat{x}_t), w\right) = 0.$$

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- Subtracting these two equations gives

$$\frac{1}{2}\frac{\partial}{\partial \tau}\|u\|_J^2 = \frac{1}{2}\frac{d}{d\tau}\int_{\Omega_h} u^2 J = 0.$$

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Energy conservation (WADG)

- In DG methods:

$$u = \boldsymbol{M}_J^{-1}\boldsymbol{M}\,(uJ) \iff (u, vJ) = (uJ, v)$$

- In WADG methods:

$$u = \boldsymbol{M}^{-1}\boldsymbol{M}_{1/J}\boldsymbol{M}^{-1}\boldsymbol{M}\,(uJ) \iff (u, v) = \left(\frac{uJ}{J}, v\right)$$

- Introduce intermediate variable $\tilde{u} \notin P^N$

$$\tilde{u} = \frac{uJ}{J} \quad \Longrightarrow \quad u = \Pi_N \tilde{u}.$$

- Take

$$v = u, \qquad w = \frac{1}{2}\Pi_N\left(\tilde{u}^2\right)$$

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- Take

$$v = u, \qquad w = \frac{1}{2}\Pi_N\left(\tilde{u}^2\right)$$

Energy conservation

Theorem (Standard DG)

The skew-symmetric ALE-DG formulation using the standard DG method is energy conservative in the sense that

$$\frac{1}{2}\frac{\partial}{\partial \tau}||u||_J^2 = 0. \quad \Longrightarrow \quad \left| ||u(\cdot, T)||_J^2 - ||u(\cdot, 0)||_J^2 \right| = 0.$$

Theorem (WADG)

The skew-symmetric ALE-DG formulation using the WADG method has an upper bound for the energy variation given by

$$\left| ||uJ(\cdot, T)||_{1/J}^2 - ||uJ(\cdot, 0)||_{1/J}^2 \right| \leq Ch^{2N+2},$$

for fixed T and sufficiently regular solution $u(x,t)$.

ALE-DG for wave propagation

- ALE system of the acoustic wave equation:

$$
\begin{aligned}
\frac{d\boldsymbol{q}J}{d\tau} + \frac{\partial}{\partial \xi_1}\left(A^1\boldsymbol{q}\right) + \frac{\partial}{\partial \xi_2}\left(A^2\boldsymbol{q}\right) &= 0, \\
\frac{\partial J}{\partial \tau} + \widehat{\nabla}\cdot\left(J\widehat{x}_t\right) &= 0,
\end{aligned}
$$

where

$$
A^1 = \begin{pmatrix} \frac{\partial \xi_1}{\partial t}J & \frac{\partial \xi_1}{\partial x_1}J & \frac{\partial \xi_1}{\partial x_2}J \\ \frac{\partial \xi_1}{\partial x_1}J & \frac{\partial \xi_1}{\partial t}J & 0 \\ \frac{\partial \xi_1}{\partial x_2}J & 0 & \frac{\partial \xi_1}{\partial t}J \end{pmatrix}, \qquad A^2 = \begin{pmatrix} \frac{\partial \xi_2}{\partial t}J & \frac{\partial \xi_2}{\partial x_1}J & \frac{\partial \xi_2}{\partial x_2}J \\ \frac{\partial \xi_2}{\partial x_1}J & \frac{\partial \xi_2}{\partial t}J & 0 \\ \frac{\partial \xi_2}{\partial x_2}J & 0 & \frac{\partial \xi_2}{\partial t}J \end{pmatrix}.
$$

ALE-DG for wave propagation

- Skew-symmetric ALE-DG formulation:

$$
\begin{aligned}
\left(\frac{d\boldsymbol{q}J}{d\tau}, \boldsymbol{w}\right) = &-\frac{1}{2}\left(\frac{\partial}{\partial \xi_1}\left(A^1\boldsymbol{q}\right), \boldsymbol{w}\right) + \frac{1}{2}\left(\boldsymbol{q}, \frac{\partial}{\partial \xi_1}\left(A^1\boldsymbol{w}\right)\right) \\
&-\frac{1}{2}\left(\frac{\partial}{\partial \xi_2}\left(A^2\boldsymbol{q}\right), \boldsymbol{w}\right) + \frac{1}{2}\left(\boldsymbol{q}, \frac{\partial}{\partial \xi_2}\left(A^2\boldsymbol{w}\right)\right) \\
&-\frac{1}{2}\left(\left(\frac{\partial}{\partial \xi_1}A^1\right)\boldsymbol{q}, \boldsymbol{w}\right) - \frac{1}{2}\left(\left(\frac{\partial}{\partial \xi_2}A^2\right)\boldsymbol{q}, \boldsymbol{w}\right) \\
&-\frac{1}{2}\left\langle \boldsymbol{q}^*, A_n\boldsymbol{w}\right\rangle, \\
\left(\frac{\partial J}{d\tau}, \theta\right) = &-\left(\widehat{\nabla}\cdot\left(J\widehat{x}_t\right), \theta\right).
\end{aligned}
$$

where $A_n = A^1\widehat{n}_1 + A^2\widehat{n}_2$ and $\widehat{n} = (\widehat{n}_1, \widehat{n}_2)$ is the reference domain normal.

Dissipative penalty fluxes

- Motivated by the surface contribution $\langle \boldsymbol{q}^*, A_n \boldsymbol{w} \rangle$

$$\boldsymbol{q}^* = \boldsymbol{q}^+ - \tau_q A_n [\![\boldsymbol{q}]\!] .$$

- When mesh reduces to the stationary case

$$A_n = \begin{pmatrix} J\widehat{x}_t \cdot n & n_1 J & n_2 J \\ n_1 J & J\widehat{x}_t \cdot n & 0 \\ n_2 J & 0 & J\widehat{x}_t \cdot n \end{pmatrix} = \begin{pmatrix} 0 & n_1 & n_2 \\ n_1 & 0 & 0 \\ n_2 & 0 & 0 \end{pmatrix}$$

- Flux $\boldsymbol{q}^*$ reduces to the standard penalty flux on a fixed domain:

$$p^* = p^+ - \tau_u [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}, \qquad \boldsymbol{u}^* = \boldsymbol{u}^+ - \tau_p [\![p]\!] \boldsymbol{n}.$$

Theorem (Consistency)

The skew-symmetric ALE-DG formulation with penalty fluxes is consistent for sufficiently regular velocity.

Dissipative penalty fluxes

Theorem (Energy stability using DG methods)

The skew-symmetric ALE-DG formulation with penalty fluxes using DG method is energy stable in the following sense

$$\frac{1}{2}\frac{\partial}{\partial \tau}\left(||p||_J^2 + ||u||_J^2 + ||v||_J^2\right) = -\tau_q [\![\boldsymbol{q}]\!]^T A_n^T A_n [\![\boldsymbol{q}]\!] \leq 0.$$

Theorem (Energy stability using WADG methods)

The skew-symmetric ALE-DG formulation with penalty fluxes using WADG method is energy stable up to a term which super-converges to zero in the following sense

$$\frac{1}{2}\frac{\partial}{\partial \tau}\left(||pJ||_{1/J}^2 + ||uJ||_{1/J}^2 + ||vJ||_{1/J}^2\right) \leq C_{max} h^{2N+2} - \tau_q [\![\boldsymbol{q}]\!]^T A_n^T A_n [\![\boldsymbol{q}]\!].$$

Constant solutions on a moving mesh

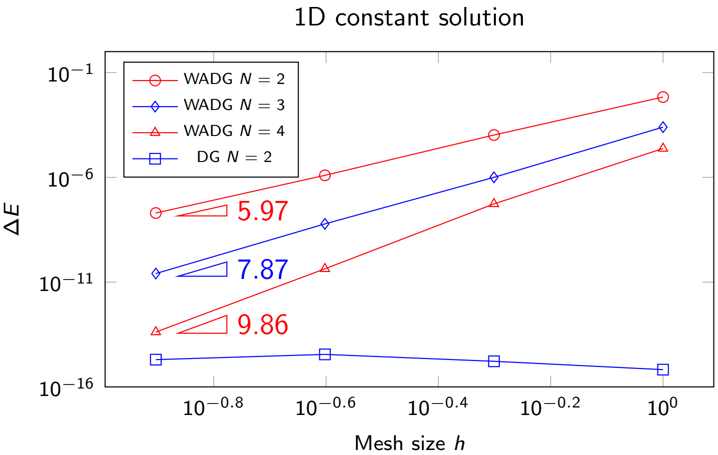
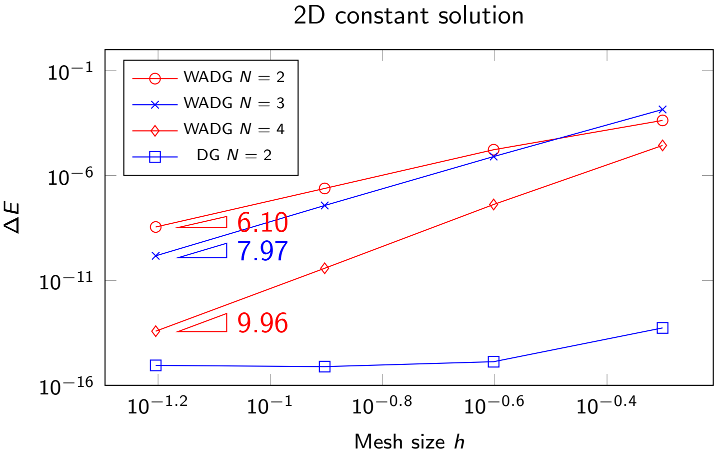


Figure: Energy variation for different orders of approximation

Bound on ΔE for ALE-WADG: $\Delta E \leq Ch^{2N+2}$

Constant solutions on a moving mesh



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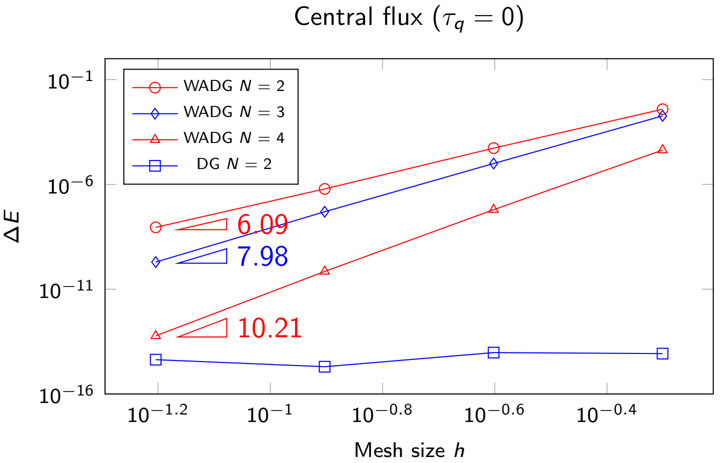
Bound on ΔE for ALE-WADG: $\Delta E \leq Ch^{2N+2}$

Energy conservation for the wave equation



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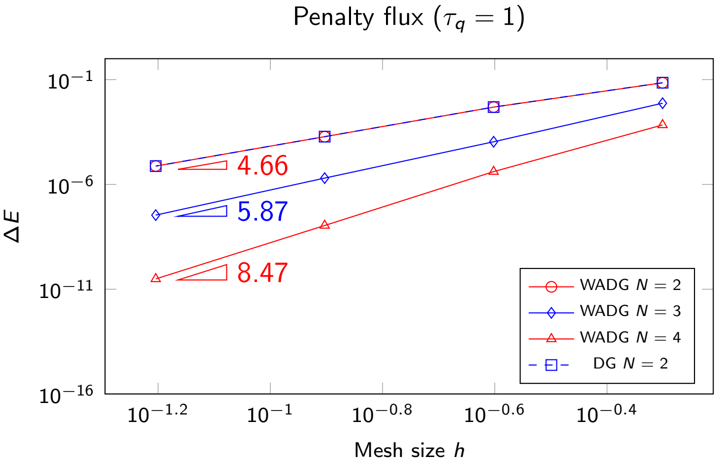
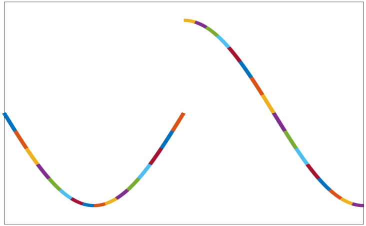


Figure: Energy variation for different orders of approximation

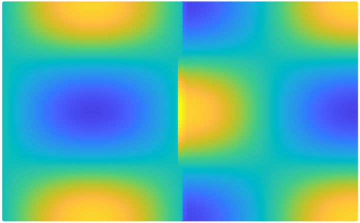
Dissipative term dominates change in energy

Energy investigation on discontinuous solutions

- Bound on ΔE does not hold for less regular solutions.
- We numerically investigate WADG for less regular solutions by considering the wave equation with discontinuous initial conditions.



(a) 1D



(b) 2D

Energy investigation on discontinuous solutions

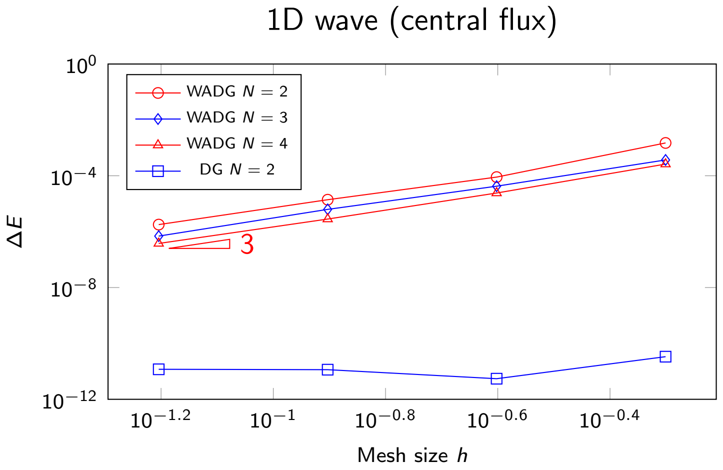


Figure: Energy variation for different orders of approximation

Energy investigation on discontinuous solutions

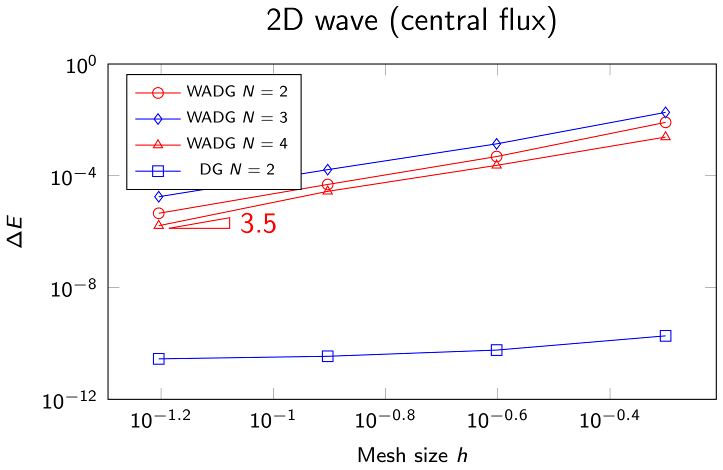


Figure: Energy variation for different orders of approximation

Energy investigation on discontinuous solutions

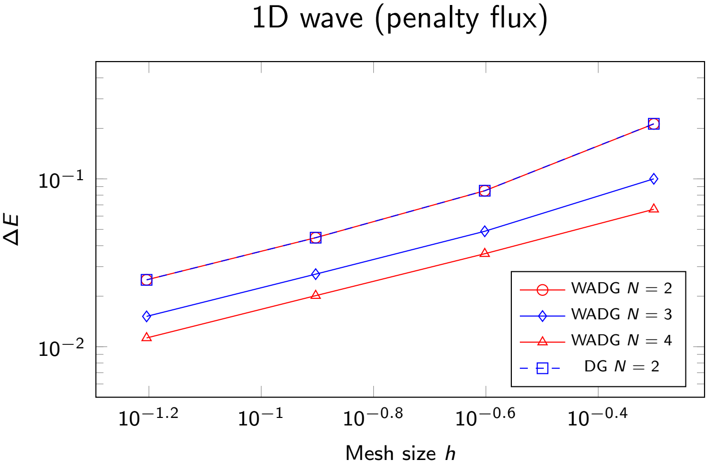
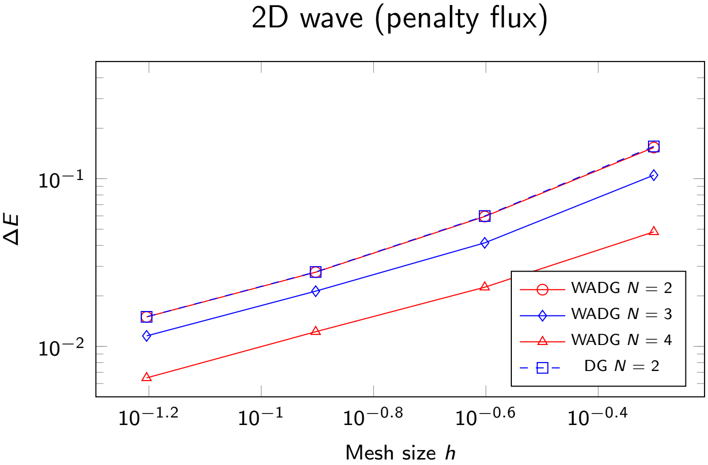


Figure: Energy variation for different orders of approximation

Energy investigation on discontinuous solutions



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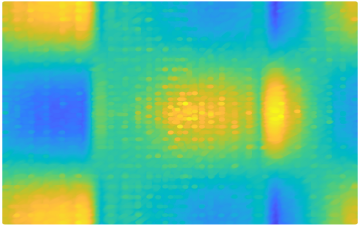
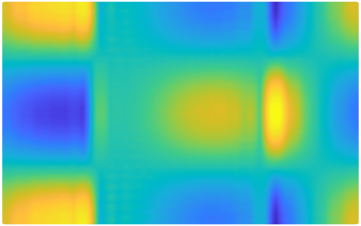
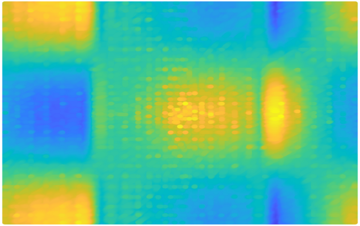
Energy investigation on discontinuous solutions



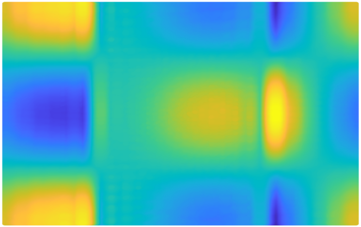
(a) DG (central flux)



(b) DG (penalty flux)

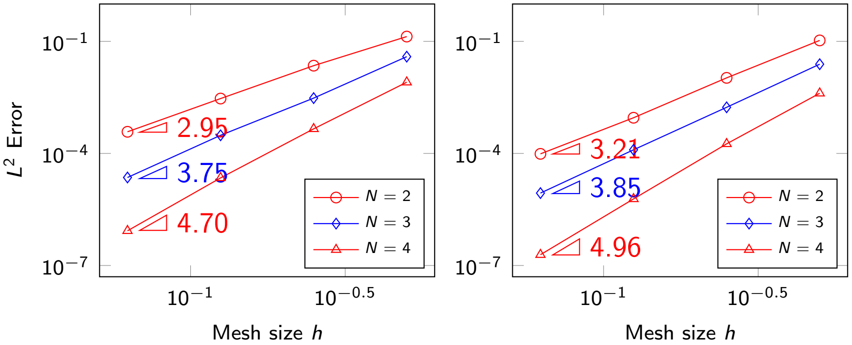


(c) WADG (central flux)



(d) WADG (penalty flux)

Convergence for the wave equation



(a) Central flux ($\tau_q = 0$) (b) Penalty flux ($\tau_q = 1$)

Figure: Convergence of L^2 errors for the acoustic wave solution

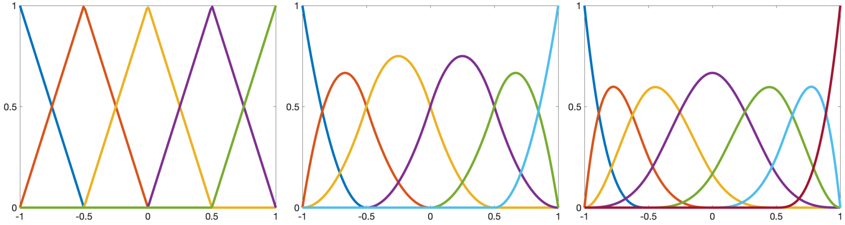
Gaussian pulse propagates on a moving mesh

(a) Moving mesh

(b) Stationary mesh

Extension to B-spline bases

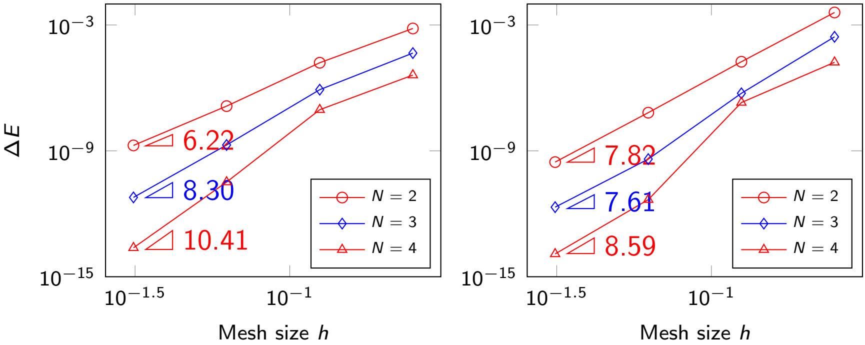
- WADG using B-spline bases:
  - B-splines form the foundations for isogeometric analysis
  - WADG recovers Kronecker structure for B-spline operators, enables efficient isogeometric analysis using explicit time-stepping



(c) $N = 1$ (d) $N = 2$ (e) $N = 3$

Figure: B-spline bases of different degrees

Energy conservation for the wave equation (B-spline)

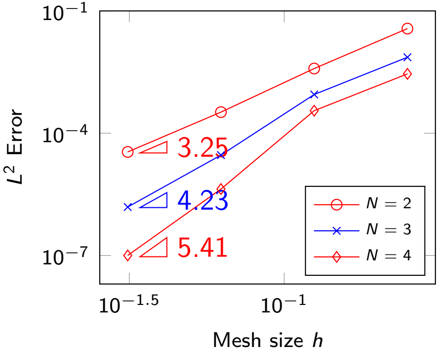


(a) Central flux ($\tau_q = 0$)

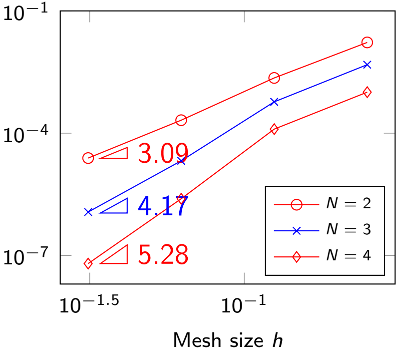
(b) Penalty flux ($\tau_q = 1$)

Figure: Energy variation for the acoustic wave solution

Convergence for the wave equation (B-spline)



(a) Central flux ($\tau_q = 0$)



(b) Penalty flux ($\tau_q = 1$)

Figure: Convergence of L^2 errors for the acoustic wave solution

Summary and acknowledgements

- We derive an ALE-DG method for wave propagation on moving curved meshes.
- Energy stability up to a term which converges to zero with the same rate as the optimal L^2 error estimate.
- The proposed method can be applied without restrictions on element type, quadrature, or choice of local approximation space.

Thank you! Questions?

Chan, Hewett, Warburton. 2016. WADG methods: wave propagation in heterogeneous media (SISC).

Guo, Chan. 2020. High order WADG methods for wave propagation on moving curved meshes.